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A NUMERICAL APPROACH TO THE CONTACT OF NOMINALLY FLAT SURFACES

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Abstract: Machined surfaces can be described by heights and wavelengths of the surface asperities that show a statistical variation. Considering that a regular wavy surface with a sinusoidal profile is the crudest model for a rough surface, studying the contact of regular wavy surfaces is a good approximation for the contact of nominally flat surfaces. Such contact problems exhibit periodicity that can be simulated with the aid of computational techniques derived for contact mechanics in the frequency domain. The displacement calculation, which is a necessary step in the resolution of the contact problem, is mathematically a convolution product that can be calculated in the frequency domain with increased computational efficiency. The displacement induced by a unit surface load can be expressed in the frequency domain by the frequency response functions, which are counterparts of the space domain solutions to half-space fundamental problems such as the Boussinesq problem. The displacement induced by a periodic pressure distribution can be computed by executing the convolution product between the frequency response function and pressure on a single period. It should be noted that the convolution calculation in the spectral domain implies that the contributions of all neighbouring pressure periods are accounted for. The need to treat numerically only a single period results in remarkable computational efficiency, allowing for high density meshes that can capture the essential features of any textured real surface. The displacement calculation promotes the solution of the contact problem by an iterative approach. The advanced method is benchmarked against existing analytical solutions for the 3D contact of surfaces possessing two-dimensional waviness. This essentially deterministic model, supported by a direct numerical solution that can be obtained for samples of real rough surfaces, presents itself as a worthy alternative to the existing statistical models for rough contact interaction.

Key words: elastic contact, convolution, fast Fourier transform, nominally flat surfaces.

1. INTRODUCTION

The design and analyses of engineering systems involve the study of the mechanical contact of materials. A contact problem aiming to derive the contact area and the pressure distribution, followed by a boundary-value problem for the stress state developed in the contacting bodies, can be formulated and solved, generally by numerical analyses, in the frame of computational contact mechanics. Theoretical analyses are mainly concerned with the study of the concentrated contact, because of its convenient experimental treatment, and due to its analytical solution, i.e., the Hertz contact.

However, nominally flat engineering surfaces are inevitably rough, and surface irregularities may perturb significantly the contact stresses computed from a smooth profile assumption. The contact analysis of nominally flat surfaces pioneered with the work of Greenwood and Williamson [1], who assumed a statistical distribution of asperities possessing spherical tips of the same radius, and Gaussian height distribution. Bhushan [2] reviewed the numerical models and simulation techniques for multi-asperity contacts. The analytical solutions for the stress fields due to bisinusoidal normal and shear tractions were calculated by Tripp et al. [3]. Westergaard [4], and Dundurs et al. [5], solved the problem of an elastic one-dimensional sinusoidal surface in contact with a flat surface. Johnson et al. [6] described the difficulties in the experimental study of the contact area established between two-dimensional sinusoidal surfaces, and applied a numerical method for contact investigation. Various contact solutions were obtained for the contact of wavy surfaces with elastic-perfectly plastic [7,8], or elastic-plastic [9] behaviour.

The accurate description of the real contact area between engineering surfaces requires discretization with large number of grids, which results in a prohibitive computational effort. This is especially true for the contact of nominally flat surfaces, which involves a large nominal contact area. Numerous research efforts in computational contact mechanics

[10-23] were focussed on improving the algorithmic efficiency by Fourier analysis, supported by the fast Fourier transform (FFT) algorithm for the calculation of the discrete Fourier transform.

This paper advances an FFT-assisted numerical solution to the contact of nominally flat surfaces, by using a surface area as a representative domain that is periodically extended laterally to mimic a periodic contact problem. The periodic displacement is calculated with increased computational efficiency in the Fourier transform domain due to the convolution theorem. Well-known results from the literature [24] are accurately reproduced with the newly developed computer program. The method promises to address an important topic and to assist the design of mechanical contacts with improved performance and reliability.

2. MODEL OVERVIEW AND ANALYTICAL RESULTS

Contact mechanics theory often assumes that the limiting surfaces of the contacting bodies are topographically smooth, which results in a continuous nominal contact area. This assumption is reasonable in selected contact cases such as mica that can be cleaved along atomic planes, or very soft rubber with asperities that can be easily flattened by the contact pressure. In most engineering applications however, the real contact area is discontinuous, and amounts to only a small fraction of the nominal (or apparent) contact area. This is due to the fact that the asperities of rough surfaces deform plastically at a contact pressure much higher (roughly three times according to the Tabor theory [25]) than the nominal pressure (i.e., the yield strength σ_Y of the material) needed for the body to yield as a whole. It is therefore of great interest to study the effects of the surface roughness on the results derived in contact mechanics based on the smooth surface profile assumption.

Machined surfaces exhibit a distinct lay, with ridges whose height and frequency follow a statistical variation [24]. Therefore, a regular wavy surface with a sinusoidal profile may be the simplest model for a rough surface. To remain in the frame of elasticity and small deformations, the wave amplitude Δ should be sufficiently small compared to the wavelength λ. With this assumption, the contact between a wavy surface and an elastic half-space can be analyzed by employing fundamental solutions derived for the elastic half-space.

When a solid bounded by a nominally flat surface having a two-dimensional regular orthogonal waviness of amplitude Δ_i and wavelength λ_i, $i = x, y$, is brought into contact with an elastic half-space, the gap hi between the undeformed surfaces can be expressed as

$$hi(x,y) = \Delta_x + \Delta_y - \Delta_x \cos(2\pi x/\lambda_x) - \Delta_y \cos(2\pi x/\lambda_y)\,. \quad (1)$$

According to the latter, the contacting surfaces touch in the corners of a uniform rectangular grid of side lengths λ_i, $i = x, y$, and the center of each rectangle marks a hollow of depth $2(\Delta_x + \Delta_x)$, which is the maximum gap between the two surfaces. Subsequently, the bodies are pressed into contact by a mean pressure p_m that is large enough to generate a continuous contact area, by fully compressing the wave. From geometrical considerations, the equation of the deformed surfaces yields:

$$hi(x,y) = \delta - u(x,y)\,, \quad (2)$$

where δ is the approach between datum points in the contacting bodies distant from the region of deformation (i.e., the rigid-body approach), and $u(x,y)$ is the normal elastic displacement of the surface under the generated contact pressure. According to [24], a sinusoidal surface traction produces a sinusoidal displacement with the same wavelength, and the pressure that verifies equation (2) can be expressed as:

$$p(x,y) = \overline{p} + P_x \cos(2\pi x/\lambda_x) + P_y \cos(2\pi x/\lambda_y)\,, \quad (3)$$

with $P_x = \pi E^* \Delta_x/\lambda_x$ and $P_y = \pi E^* \Delta_y/\lambda_y$, and $1/E^* = (1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2$, with E_i and ν_i, $i = 1, 2$ the Young moduli and the Poisson's ratios of the contacting materials. If furthermore the wavy surface is assumed isotropic, i.e., $\Delta_x = \Delta_y = \Delta$ and $\lambda_x = \lambda_y = \lambda$, equation (3) becomes

$$p(x,y) = p_m + P\left(\cos(2\pi x/\lambda) + \cos(2\pi y/\lambda)\right), \quad (4)$$

with $P = \pi E^* \Delta/\lambda$. Both (3) and (4) hold if and only if

$$p_m \geq P_x + P_y\,, \quad (5)$$

resulting in a continuous contact area. If condition (5) is violated, regions of contact will coexist with regions of separation. Low p_m will generate contact regions given by the Hertz framework, but other than that, numerical solutions are required [24].

The mathematical model for the frictionless contact between two bodies with known initial contact geometry can be reduced [26] to a linear system of equations by digitization of equation (2), having the discrete pressures as unknowns. The latter system is solved by the conjugate gradient method, whose rate of convergence is superlinear. As the contact area established during elastic deformation is unknown, the size of the system resulting from (2) is also unknown. The state-of-the-art solution [26] is to determine the contact area and the pressure distribution simultaneously, in an iterative manner: a trial contact area is adopted, and an associated pressure is calculated based on constraints comprising equation (2), the static force equilibrium, and the complementarity conditions. The contact area and the discrete pressure elements are adjusted until all the above restrictions are fulfilled. The condition for the application of this strategy is the existence of a method for the computation of displacement induced by arbitrary, yet known, pressure on an irregular contact area, as resulting during the iterative process. An efficient numerical method for calculation of displacement due to periodic surface tractions with arbitrary distribution inside the period, is described in the following section.

3. THE SPECTRAL CALCULATION OF CONVOLUTION

Once the elastic contacting bodies are assimilated with elastic half-spaces, results from the linear theory of elasticity can be applied. The normal displacement induced in an elastic half-space by a distribution of pressure $p(\xi,\eta)$ acting on the half-space boundary can be expressed as [24]:

$$u(x,y)=\frac{1-\nu^2}{\pi E}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{p(\xi,\eta)}{\sqrt{(x-\xi)^2+(y-\eta)^2}}d\xi d\eta \,. \quad (6)$$

The latter integral is mathematically a continuous linear convolution between pressure and a Green's function $G(x,y)$, i.e., the normal displacement induced in an elastic half-space by a unit point force acting normally on the half-space boundary in origin of the coordinate system. The closed-form expression of the latter was obtained by Boussinesq, cited in [24]:

$$G(x,y)=\frac{1-\nu^2}{\pi E}\frac{1}{\sqrt{x^2+y^2}} \,. \quad (7)$$

The convolution product of two functions, denoted by the symbol "$\otimes$", is used in digital signal processing to derive an output signal based on the input signal and its unit impulse response, which in case of (6) is exactly the Green's function (7). Equation (6) can be written as a continuous convolution:

$$u(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p(\xi,\eta)G(x-\xi,y-\eta)d\xi d\eta = p(x,y)\otimes G(x,y)\,. \quad (8)$$

The continuous convolution theorem postulates that, by applying the Fourier transform to both sides of (8), an analogue relation between the Fourier transform of the convolution members is obtained, in which the convolution operator is substituted by term-by-term multiplication:

$$\tilde{\tilde{u}}(m,n)=\tilde{\tilde{p}}(m,n)\cdot\tilde{\tilde{G}}(m,n)\,. \quad (9)$$

Here, the double "~" symbol is used to denote continuous double Fourier transform (with respect to the x and y directions), and m, n, are the frequency coordinates corresponding to the x and y directions, respectively. The convolution product in the space domain is thus replaced by simple multiplication of complex numbers in the frequency domain. Moreover, the Fourier transform of (7) can be obtained in closed-form:

$$\tilde{\tilde{G}}(m,n)=\frac{2(1-\nu^2)}{E\sqrt{m^2+n^2}}\,. \quad (10)$$

The spectral displacement in (10) can be digitized at a series of discrete frequencies comprised in a frequency domain that is mathematically related to the problem space domain. The discrete counterpart of (8) can be written as the product of discrete series:

$$u_{i,j}=\sum_{\ell=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}p_{k,\ell}G_{i-k,j-\ell}\,. \quad (11)$$

However, a numerical solution cannot cover an infinite problem domain. Presuming the discretization of a rectangular domain expected to encompass the contact area, all problem parameters should be calculated in N uniformly distributed control points, $N=N_xN_y$, with N_i, $i=x,y$, indicating the number of grids in the corresponding direction. In between, a piece-wise constant distribution is assumed for simplicity. The order of computation of (11) is $O(N^2)$ by direct multi-summation. On the other hand, (9) is an $O(N)$ operation, and the transfer to and from the frequency domain by FFT is of $O(N\log_2 N)$. Giving the remarkable efficiency of the FFT algorithm in the computation of the discrete Fourier transform, both direct and inverse, it is convenient to execute the convolution calculation (11) in the frequency domain. Moreover, when a discrete series is transferred to the frequency domain, it is tacitly assumed that the series comes as a period from a periodic distribution.

This periodic feature of the Fourier transform suggests a method for the numerical study of rough contacts other than the statistical modelling pioneered by Greenwood and Williamson [1]. Deterministic approach based on measured topography can be applied in the following manner: the numerical data (the height information) measured for a representative surface patch is periodically extended in both lateral directions to create a large rough half-space, and the contact analyses is performed on one period only. In this manner, a contact problem with an infinite domain such as the contact of nominally flat but rough surfaces, can be treated as a periodic contact problem. As the frequency response function (10) exists in closed form, and the periodic pressure distribution can be transformed into a Fourier series, the convolution theorem (9) should be applied to solve the periodic contact problem, as described in detail in the following section.

4. ALGORITHM DESCRIPTION

The proposed contact algorithm employs a periodic multiplication strategy of a representative domain. All contact parameters, such as initial contact geometry, pressure, and displacement, are thus assumed periodic, with the same period equal to the domain extent. All computations are performed on a period only, but the application of the Fourier convolution technique guarantees that the results are valid for the infinite periodic domain. In case of the wavy isotropic surface of separation described by (1), it is convenient to choose as target domain a rectangle of side lengths λ_x and λ_y, reported to a coordinate system with its origin in the center of the rectangle. The rectangle is then divided into $N = N_x N_y$ elementary patches. The contact geometry data should be available in such a way that a representative value can be chosen for each elementary patch. In case of a closed-form relation or computer generated topography, the value at the center of each patch can be considered. The pressure distribution is digitized into a matrix of elements $p_{i,j}$, with $i = 1 \ldots N_x$ and $j = 1 \ldots N_y$. The double (i.e., with respect to both x and y directions) FFT of the latter is then computed, giving a matrix of the same size, but having complex elements: $\hat{\hat{p}} = FFT(p)$. In this manner, one term of the convolution is obtained in discrete form in the frequency domain, and the assumption of periodicity is present via FFT.

For the second convolution member, a frequency domain associated to the chosen space domain $[-\lambda_x/2, \lambda_x/2] \times [-\lambda_y/2, \lambda_y/2]$ is first discretized into $N_x \times N_y$ frequencies:

$$m_i = 2\pi(i - N_x/2 - 1)/\lambda_x, i = 1 \ldots N_x; \quad n_j = 2\pi(i - N_y/2 - 1)/\lambda_y, j = 1 \ldots N_y, \tag{12}$$

and the frequency response function (10) is calculated in these frequencies:

$$\hat{\hat{G}}_{i,j} = \tilde{\tilde{G}}(m_i, n_j), i = 1 \ldots N_x, j = 1 \ldots N_y. \tag{13}$$

An additional difficulty arises at this point, as the origin of the frequency domain is a member of the discretization (12), and there the frequency response function (10) is singular. This underdetermination was circumvented by Nogi and Kato [11], who observed that $\tilde{\tilde{G}}$ was numerically integrable on a vicinity of origin. Therefore, the value that locally describes best the function behaviour was chosen as the mean value over the patch centred in origin:

$$\hat{\hat{G}}_{\frac{N_x}{2}+1, \frac{N_y}{2}+1} = \tilde{\tilde{G}}(0,0) \cong \frac{\lambda_x \lambda_y}{4\pi^2} \int_{-\pi/\lambda_y}^{\pi/\lambda_y} \int_{-\pi/\lambda_x}^{\pi/\lambda_x} \tilde{\tilde{G}}(\xi, \eta) d\xi d\eta. \tag{14}$$

The calculated matrix $\hat{\hat{G}}$ is then rearranged in wrap-around order, as depicted in Table 1. The term-by-term product between the discrete matrices of complex elements is computed according to (9), giving the spectral counterpart of displacement. The space domain displacement is eventually recovered in the space domain by inverse FFT, giving the algorithm output, i.e., the periodic displacement due to periodic pressure distribution having as period the considered domain with N elementary patches.

$$u = IFFT(\hat{\hat{u}}) = IFFT(\hat{\hat{p}} \cdot \hat{\hat{G}}). \tag{15}$$

Table 1. Rearrangement of discrete samples of the frequency response function

Original position	Rearrangement in wrap-around order
$[1, N_x/2] \times [1, N_y/2]$	$[N_x/2+1, N_x] \times [N_y/2+1, N_y]$
$[N_x/2+1, N_x] \times [N_y/2+1, N_y]$	$[1, N_x/2] \times [1, N_y/2]$
$[1, N_x/2] \times [N_y/2+1, N_y]$	$[N_x/2+1, N_x] \times [1, N_y/2]$
$[N_x/2+1, N_x] \times [1, N_y/2]$	$[1, N_x/2] \times [N_y/2+1, N_y]$

5. RESULTS AND DISCUSSIONS

A Hertz point contact scenario is first considered for comparison, having a central pressure p_H and a contact radius a_H. The related (non-periodic) displacement $u^{(Hertz)}$ in radial coordinates [24]

$$u^{(Hertz)}(r) = \pi p_H \left(2a_H^2 - r^2\right) / \left(2E^* a_H\right), \quad |r| < a_H, \quad (16)$$

is compared in Figure 1 with the periodic displacement due to a Hertz-type pressure, calculated with the newly proposed algorithm. The representative domain chosen in the calculation is a square of side lengths $2a_H$. The periodic displacement in Figure 1 is thus due to a periodic pressure obtained by lateral multiplication of a semi-ellipsoidal Hertz pressure. As expected, the displacement at the periphery is underestimated when a non-periodic scenario is employed for a periodic problem. Dimensionless displacement is defined as ratio to Hertz normal approach, δ_H, which is different from that for the periodic problem. The accuracy of the calculation for the rigid-body approach by the proposed algorithm is conditioned by the precise computation of quadrature in (14).

Figure 2 depicts the 3D numerical pressure obtained on a single period in the contact involving an isotropic wavy surface of the form (1), with $\Delta_x = \Delta_y = \Delta$, $\lambda_x = \lambda_y = \lambda$, and $\bar{p} = 2P$. The applied normal contact load was $p_m \lambda^2$. It should be noted that this distribution extends periodically in the lateral directions. Dimensionless pressure is defined as ratio to the mean pressure p_m, and dimensionless coordinates as ratios to the wavelength λ, which is also the side length of the square representative domain. One can see that the pressure is positive everywhere, so a continuous contact area is established, in agreement with the theoretical framework (1)-(5).

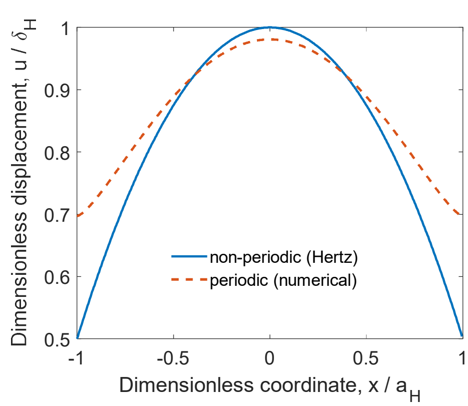


Fig. 1. Periodic vs. non-periodic displacement calculation in a Hertz point contact

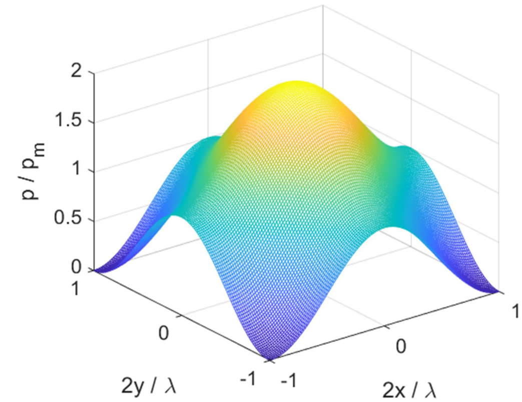
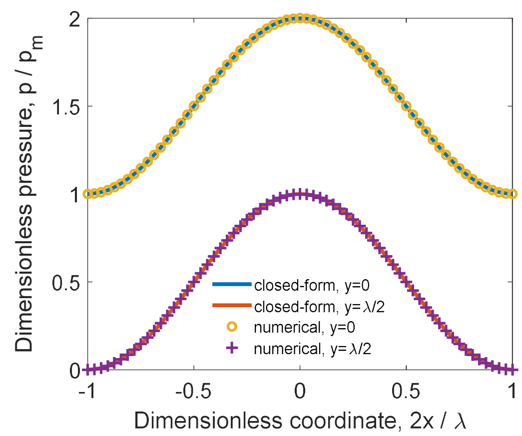
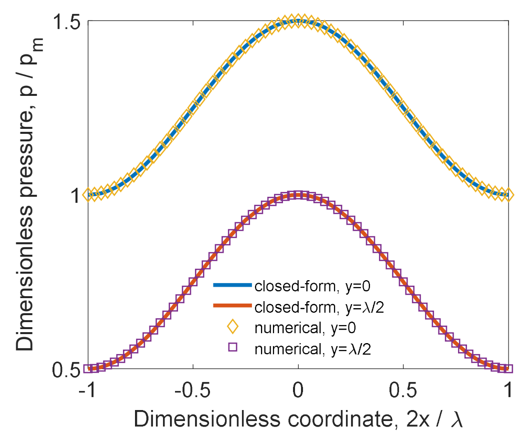


Fig. 2. A period of the 3D pressure distribution due to a wavy isotropic surface



(a)



(b)

Fig. 3. Pressure profiles due to a wavy isotropic surface: a) $\bar{p} = 2P$; b) $\bar{p} = 4P$.

Figure 3 compares the pressure distribution predicted by the newly advanced computer program with the analytical expression (4). The plot in Figure 3(a) correspond to the case depicted in Figure 2, whereas in Figure 3(b) the amplitude of the waviness is kept constant, but the mean pressure is doubled. The good agreement obtained for both cases suggests the method precision in the derivation of pressure distribution arising in periodic contact problems.

The contact of nominally flat surfaces can be further treated with the proposed algorithm in the following manner: (1) obtain the surface topography data by 3D imaging devices, such as optical profilometers; (2) select a representative window as a problem period, and (3) solve the contact problem for the selected single period. The results obtained for the latter period implicitly take into account the periodic nature of the contact problem.

6. CONCLUSIONS

The statistical models for the contact of nominally flat surfaces may have limitations because the real topography is not accounted for. A deterministic model, supported by a direct numerical solution obtained for samples of real rough surfaces, presents itself as a worthy alternative, but the surface digitization and solution can only be performed over a finite area.

The strategy proposed in this paper is to perform the contact analysis on a representative domain whose periodic lateral multiplication replicate the real surface in a satisfactory manner. The pressure-displacement dependence is handled in the Fourier transform domain, and the computed displacement response is periodical with a period equal to the domain window.

Besides periodicity proper handling, the proposed computational technique has the advantage of performing the convolution operation in the frequency domain, as a term-by-term product between the Fourier transforms of the convolution members. This reduces significantly the order of operations for the displacement computation, especially when the highly efficient FFT algorithm is used for the discrete Fourier transform.

The newly advanced computer program reproduces accurately results from the literature concerning the periodic contact of wavy surfaces, although the contact solution was performed on a single period.

The method should be used with care when the rigid-body approach between the contacting bodies is needed, due to the singularity of the frequency response functions in the origin of the frequency domain. Whereas the numerical treatment of the latter sample may introduce algorithm instability, the solution of a contact problem in terms of pressure and contact area, only requires relative displacements and therefore it is not affected by this underdetermination. A robust numerical solution with increased computational efficiency for the contact of nominally flat but rough surfaces is thus advanced.

7. ACKNOWLEDGEMENT

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